THE SINGLE PERIOD INVENTORY MODEL: ORIGINS, SOLUTIONS, VARIATIONS, AND APPLICATIONS

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THESIS

THE SINGLE PERIOD INVENTORY MODEL:
ORIGINS, SOLUTIONS, VARIATIONS,
AND APPLICATIONS

by

Junichi Masuda

September 1977

Thesis Advisor: G.F. Lindsay

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

The Single Period Inventory Model: Origins, Solutions, Variations, and Applications

5. TYPE OF REPORT & PERIOD COVERED

Master's Thesis; September 1977

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

Junichi Masuda

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Naval Postgraduate School
Monterey, California 93940

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

Naval Postgraduate School
Monterey, California 93940

12. REPORT DATE

September 1977

13. NUMBER OF PAGES

61

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Single Period Inventory Model

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
(Page 1) S/N 0102-014-6601 |

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(20. ABSTRACT Continued)

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The Single Period Inventory Model: Origins, Solutions, Variations, and Applications

by

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Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1977

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The classic newspaper boy or single period inventory problem is reviewed, and its origins and development during the past 30 years are traced. The review reveals several variations for the classic problem, both in cost structure and in the decision principles involved. The critical role of costs and demand functions to solutions is examined. A large variety of potential applications for this model is summarized from the literature. Also, optimal solutions are derived for the constant surplus cost case and the exponential cost case.

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ACKNOWLEDGEMENTS

I gratefully acknowledge the considerable time and effort that my thesis advisor, Professor G. F. Lindsay from the U.S. Naval Postgraduate School Department of Operations Research, expended while assisting me in my thesis effort.

I. INTRODUCTION

This thesis is devoted to the origins and development of a single-period inventory problem best known as a newspaper boy problem or a Christmas tree problem. In the past, a variety of names have been used, such as the "News boy problem", [1] "Newspaper boy problem", [2] "Newspaper vendor problem", [3] "Newsvendor problem", [4] "Christmas tree problem", [5] and so on. We shall call it "Newspaper boy problem" in this paper.

Although Morse and Kimball are considered an original source of this problem [1], they did not provide an analytic solution. In this thesis we shall trace the history of the newspaper boy problem, and the development of the very simple and useful analytical solution. We shall also show an expansion of the cost function, and describe applications of this problem in the real world. In doing this, we hope to assist the reader in identifying other applications for this rather powerful part of inventory theory.

First of all we will review the well-known newspaper boy problem in Chapter II, and show some fundamental solutions. In Chapter III, we shall trace the origins and development of the newspaper boy problem and the derived analytic solution. In Chapter IV we will show a variation of this problem by considering a constant for surplus cost and linear function for shortage cost, in both the discrete and continuous cases.

A review of real world applications follows consideration for the demand function in Chapter V. It is felt that this demand function is a very important consideration when we approach real world situations. Conclusions and suggestions for further work are given in Chapter VI.

II. A BRIEF HISTORY OF INVENTORY MODELS

Many people have been involved in the construction and use of models for inventory management, and many papers have appeared since 1915. Our concern centers upon who first provided the analytic solution for the well-known newspaper boy problem. In this chapter we shall trace it over the years through periodical journals and many publications.

In introducing the historical development of the newspaper boy problem, it is useful to begin with the origins of earlier general inventory models.

Inventory control has been recognized as an important problem in the business world for a great many years. The earliest known analysis of an inventory problem was by F.W. Harris in 1915 [2],[5],[7]. He is considered to be the person who first published the simplest type of "lot size" calculations, and his is the oldest and most widely known inventory model.

His basic formula is derived as follows:

Q = Purchase quantity,

C = Unit cost,

λ = Expected yearly sales,

A = Procurement expense involved in making one order,

I = Inventory carrying cost,

$f(Q)$ = Total annual cost,

and

$$f(Q) = \frac{A\lambda}{Q} + \frac{ICQ}{2} \quad .$$

In order to determine the value of Q which minimizes the total annual cost f(Q), we take the derivative of the total annual cost function with respect to Q and set it equal to zero. The following result is obtained:

$$Q = \frac{2A\lambda}{IC} \quad .$$

This result gives the Q which yields the absolute minimum of the total cost f(Q). It is sometimes called the lot size formula, the economic order quantity, the square root formula, the Harris formula, or the Wilson formula since it was also derived by R.H. Wilson as an integral part of the inventory control scheme which he sold to many organizations.

The problem of determining optimal inventory levels occupied the attention of a number of investigators in the early 1920's, resulting in the publication during 1925, 1926, and 1927 of several papaers. Whitin [7] particularly mentions "How to Determine Economic Manufacturing Quantities" by Benjamin Cooper, "Practical Lot Quantity Formula" by George F. Mellen, and "How to Maintain Proper Inventory Control" by H.S. Owen. All utilize essentially the same approach for determining economic purchase quantities. In particular, their approach involved a solution of the purchase

quantity problem combined with a method of determining safety allowances, so as to "completely" determine the level of inventory. [7]

A new and important dimension in inventory level determination was reported by Arrow, Harris and Marschak in "Optimal Inventory Policy" in July 1951. [8] They studied uncertainty models — a static and dynamic one — in which the demand flow is a random variable with a known probability distribution. They determined the best maximum stock and the best reordering point as functions of the demand distribution, the cost of making an order, and the penalties for stock depletion.

Since their fundamental work, there has been considerable study of inventory systems. Further advanced research was accomplished by Dvoretsky, Kiefer, and Wolfowitz in their two papers, "The Inventory Problem," which were published in April 1952 and July 1952. [9] They give proofs for the existence and uniqueness of resolutions of the inventory problem for an extremely wide range of situations, including the many-commodity and cascading-sequence cases. Since 1952, more and more studies have contributed to inventory theory, and numerous publications have been devoted to the study of inventory systems. [3],[10],[11]

Now we shall look at the newspaper boy problem which is our primary interest here. Morse and Kimbal showed only an idea of the problem in their famous book Methods of Operations Research in 1950. [1]

Although their book describes the problem, there does not appear to be evidence of a published analytic solution at this time. Interest in stochastic inventory models was growing, however, and a stochastic version of the simple lot size model was developed by Whitin in <u>The Theory of Inventory Management</u> which in 1953 was the first book which dealt in any detail with stochastic inventory models. His bibliography marvelously contains 180 entries and covers the period from 1922 to 1951. [7]

Shortly thereafter, Whitin and Youngs derived mathematically the solution to the newspaper boy problem in "A Method for Calculating Optimal Inventory Levels and Delivery Time" which was reported in the <u>Naval Research Logistic Quarterly</u> (under the auspices of the Sloan Research Fund, School of Industrial Management, M.I.T. and Indiana University) in September, 1955. [12] Whitin later included the newspaper boy solution in the new edition of his book (1957), which has 43 additional bibliographic entries which cover publications up to 1956. [7]

We will realize that the newspaper boy analytic solution is a powerful mathematical tool if we look at the papers "The Inventory Problem," written by Faderman, Littauer, and Weiss in 1953 [3] and based upon the two earlier papers of the same title. They solved an actual stochastic inventory problem of the Navy, as follows:

> There are certain rather expensive items (some costing over $100,000 each) known as "insurance spares" which are generally procured at the time

a new class of ships is under construction. These spares are bought even though it is known that it is very unlikely that any of them will ever be needed and that they cannot be used on any ship except those of that particular class. They are procured in order to provide insurance against the rather serious loss which would be suffered if one of these spares were not available when needed. Also, the initial procurement of these spares is intended to be the only procurement during the lifetime of the ships of that class, because it is extremely difficult and costly to procure these spares at a later date. The present policy is to order quantities of these spares according to the following schedule:

Total # of items installed	# of spares ordered
1- 4	1
5- 50	2
51-100	3
over 100	4

The problem here, of course, is to determine the number of spares to order. Their presentation of the optimal solution required a full printed page of step by step details. Naddor later pointed out that this problem could be solved by the analytic solution to the newspaper boy problem, and in his 1966 book Inventory Systems [2] he used only a few lines to show the optimal solution, thus demonstrating the contribution of Whitin's work.

A more recent application of the newspaper boy model is "A Class of One-Period Inventory Models," by Stan Fromovitz in 1964.[13] He defined a stock-out penalty function instead of considering shortage costs, and imposed a constraint on it. He exhibits the general randomized, optimal policy and gives a graphical and numerical method of

calculating it. He also discusses how the stock-out penalty function is related to shortage cost. This approach becomes very useful when a shortage cost is difficult to measure.

Many articles on these subjects still appear regularly in Operations Research, Management Science, Econometrica, Industrial Engineering and AIIE Transactions, Naval Research Logistic Quarterly, and other journals. Later, we will describe some applications of this theory to real world problems involving shortage and surplus costs.

III. WHAT IS A SINGLE-PERIOD INVENTORY PROBLEM?

To begin our discussion of the single-period problem often referred to as the newspaper boy problem, we shall in this chapter introduce notation for the newspaper boy problem, discuss the characteristics of this model, and give the famous analytic solution.

The inventory models in this paper all deal with a single commodity and a single time period. The quantities left at the end of the period never carry over to the next period. These are essential characteristics of the model. A rather wide variety of real world inventory problems including the stocking of spare parts, perishable items, style goods, and special seasonal items offer practical examples of the sort of situations to be studied here.

These inventory systems usually have the following three kinds of costs which are important:

(1) Carrying Costs,

(2) Shortage Costs,

and

(3) Replenishing Costs.

The first kind of cost, that of carrying inventory, may include such items as rent, insurance, depreciation, loss of interest on capital invested, etc. There are costs which may be associated with the excess of supply over demand and charged after demand for the period has occurred. We often call the sum of these carrying costs, surplus costs.

The second kind of cost, shortage cost, is associated with a failure to satisfy demand immediately when it occurs. The third kind of cost, replenishing cost, is the cost of machine set-ups for production runs, of preparing orders, of handling shipments, etc. An inventory system has been defined as a system in which two or three of these kinds of costs are subject to control. We will consider the newspaper boy or Christmas tree problem as an inventory system in which carrying cost and shortage costs are subject to control.

An inventory problem is a problem of making optimal decisions with respect to an inventory system. In other words, an inventory problem is concerned with the making of decisions that minimize the total expected cost of an inventory system. The general single-period inventory model in this paper is often referred to as the newspaper boy problem or the Christmas tree problem, since it can be phrased as a problem of deciding how many trees a dealer in Christmas trees should purchase for the season, or how many newspapers a boy should buy on a given day for his corner newsstand.

Let us illustrate the newspaper boy problem. Assume that a newspaper vendor must order his papers at the beginning of the day, and that he has no opportunity to reorder later in the day if he needs more papers. Thus in a given day he has only one opportunity to place an order. Suppose a newspaper vendor buys I papers from the publisher for 20 cents a copy, sells D copies for 25 cents, and resells the unsold copies

to the publisher for 17 cents a copy. He makes 5 cents profit on each of the D papers he sells and loses 3 cents on each of the (I - D) papers he returns.

This newspaper boy problem is a special case of a more general model which can be formulated as follows.

Let

I = The inventory level at the beginning of the period,

D = A random variable representing demand for the item during the period,

I_o = The optimal inventory level,

C_1 = Surplus cost per unit remaining at the end of the period,

C_2 = Shortage cost per unit of unfilled demand at the end of period,

$p(D)$ = Probability that D units will be demanded during the period,

$F(D)$ = Distribution function for $p(D)$,

and

$E(I)$ = The total expected cost associated with a inventory level of I units.

The cost equation is

$$\text{Cost} = \begin{cases} C_1(I - D) & 0 \leq D \leq I \quad , \\ C_2(D - I) & I < D \quad . \end{cases}$$

We will present the optimal solution under risk and uncertainty using the standard cost expression in the next two sections.

A. OPTIMAL SOLUTIONS UNDER RISK

If we know the demand density function $p(D)$ and thus the distribution function $F(D)$, we call the problem as being under conditions of risk. When demand is a discrete random variable, we determine the optimal inventory level I_o by minimizing the total expected cost function

$$E(I) = \sum_{D=0}^{I} C_1 (I - D) p(D) + \sum_{D=I+1}^{\infty} C_2 (D - I) p(D) .$$

Two necessary conditions for minimum at I_o are

(1) $E(I_o + 1) - E(I_o) \geq 0$,

and

(2) $E(I_o - 1) - E(I_o) \geq 0$.

Using these conditions, it is possible to derive the following famous analytic solution for the minimum cost inventory level I_o. [14]

$$F(I_o - 1) < \frac{C_2}{C_1 + C_2} < F(I_o) . \tag{1}$$

We shall illustrate the use of this result with the Navy Spares application which was described in Chapter I. Suppose spares cost (C_1) \$100,000 each and a loss (C_2) of \$10,000,000 is suffered for each spare that is needed when there is no available stock. From (1) we have

$$F(I_o - 1) < 0.9900 < F(I_o) \quad .$$

We wish to find the optimal level I_o which satisfies the above inequality conditions. Suppose the probabilities for spares required are as in the following table.

TABLE I

Demand Distribution for Spares

D	P(D)	F(D)
0	0.9488	0.9488
1	0.0400	0.9888
2	0.0100	0.9988
3	0.0010	0.9998
4	0.0002	1.0000

From Table I we find

$$F(1) = 0.9888$$

and

$$F(2) = 0.9988 \text{ , therefore, } I_o = 2 \text{ ,}$$

that is, two spare parts should be procured.

If demand is assumed to be a continuous random variable, then we wish to find the optimal inventory level I_o which minimizes the total expected cost

$$E(I) = C_1 \int_0^I (I-D)P(D)dD + C_2 \int_I^\infty (D-I)p(D)dD \quad .$$

Taking the derivative with respect to I and setting the results equal to zero, we find that a necessary condition for a relative maximum or relative minimum at I_o is

$$F(I_o) = \frac{C_2}{C_1 + C_2} \quad . \tag{2}$$

Since $\dfrac{d^2E(I)}{dI^2} = (C_1 + C_2)p(I_o) \geq 0$, we have a minimum at $I = I_o$.

B. ASPIRATION LEVEL SOLUTIONS

An aspiration level is simply some level of profit which the decision maker desires to attain, or some level of cost which he desires not to exceed. We might wish to fix the inventory level so that his chance of not achieving his aspiration level is minimized. To do this the sum of the probability that

$C_1(I-D) > A$, and probability that

$C_2(D-I) > A$,

or

$$\int_{0}^{I-\frac{A}{C_1}} p(D)dD + \int_{I+\frac{A}{C_2}}^{\infty} p(D)dD \quad \text{is to be minimized.}$$

Morris [14] shows that this sum will be minimized for a value I_o such that the conditions

$$p(I_o - \frac{A}{C_1}) = p(I_o + \frac{A}{C_2})$$

is realized.

C. OPTIMAL SOLUTIONS UNDER UNCERTAINTY

When we do not know the demand density or distribution function, we deal with the problem as one under conditions of uncertainty. The Laplace principle for uncertainty is easily applied in this decision for it simply suggests that the density function of demand be taken as a uniform density function. In this problem it would be taken to be uniform over the range 0 to D_{max}. Thus $P(D) = \frac{1}{D_{max}}$, $F(D) = \frac{D}{D_{max}}$, and the optimal solution for this discrete case may be found from (1) to be

$$I_o < \frac{C_1(D_{max} + 1)}{C_1 + C_2} < I_o + 1 \quad .$$

For the continuous case, we use the result (2) obtained before. Here,

$$\int_0^{I_o} p(D)dD = F(I_o) = \frac{I_o}{D_{max}} = \frac{C_2}{C_1 + C_2},$$

and thus

$$I_o = \frac{C_2 D_{max}}{C_1 + C_2}.$$

To minimax regret, when demand is discrete, Morris [14] shows that the value of I_o which satisfies the following inequality

$$I_o < \frac{C_1(D_{max} + 1)}{C_1 + C_2} < I_o + 1$$

gives us the optimal inventory level, while when demand is continuous, the condition is

$$I_o = \frac{C_1(D_{max})}{C_1 + C_2}.$$

IV. A SINGLE-PERIOD MODEL WITH CONSTANT COST FOR SURPLUS

Single-period models having the standard linear cost function were reviewed in Chapter II. In looking at real world problems, we may have to consider a variety of cost functions, such as quadratic cost functions, constant cost functions and so on. Solutions for the newspaper boy problem with quadratic cost functions, and with linear cost functions for surplus and constant cost for shortage have been investigated in previous theses [15],[16]. In this chapter, we will study the newspaper boy problem with a constant cost for surplus and linear cost function for shortage, as an extension of a standard single-period model. The cost equation is

$$\text{Cost} = \begin{cases} K & D = 0,1,2,\ldots,I \\ C_2(D-I) & D = I+1, I+2, \ldots, D_{max} \end{cases} .$$

A situation for which this cost function might be appropriate can be explained by overtime costs. Overtime involves wage payments at an hourly rate that usually is 50 per cent higher than regular time. Idle time is a waste of labor time that is paid for in the regular payroll, but is not used for production activities. Some production level is supposed to be the optimal level I_o for a workforce on regular time from 8:00 AM to 5:00 PM for a given period,

and production cannot be extended to the next period. Suppose the company produces less product than I_o since the company does not know the optimal inventory level I_o. The company must still pay for the regular payroll K. This means a waste of labor time. On the other hand if the company produces more than I_o, the employees will be forced to do overtime work. This means the company must pay higher wages than for regular time. The problem for this cost function is to determine the optimal inventory level I_o.

A. MINIMIZING EXPECTED COST WHEN DEMAND IS DISCRETE

When the demand is discrete, the expected total cost function is

$$E(I) = K \sum_{D=0}^{I} P(D) + \sum_{D=I+1}^{\infty} C_2(D - I)p(D) . \qquad (3)$$

Substituting I+1 instead of I in (3), we obtain

$$E(I+1) = K \sum_{D=0}^{I} p(D) + Kp(I+1) + \sum_{D=I+2}^{\infty} C_2(D-I-1)p(D) .$$

Similarly, substituting I-1 instead of I in (3), we obtain

$$E(I-1) = K \sum_{D=0}^{I} p(D) - Kp(I) + \sum_{D=I}^{D} C_2(D-I+1)p(D).$$

Substituting these expressions into the conditions for a minimum,

$$E(I_o+1) - E(I_o) > 0 ,$$

and

$$E(I_o-1) - E(I_o) > 0 ,$$

we obtain

$$\frac{1 - F(I_o)}{p(I_o+1)} < \frac{K}{C_2} ,$$

and

$$\frac{1 - F(I_o-1)}{p(I_o)} < \frac{K}{C_2} .$$

From these, the decision rule for a minimum at I_o is

$$\frac{1 - F(I_o)}{P(I_o+1)} < \frac{K}{C^2} < \frac{1 - F(I_o-1)}{p(I_o)} . \tag{4}$$

Hence, we can find the optimal inventory level I_o which satisfies the double inequality (4) above.

To illustrate this, suppose demand has a Poisson distribution with $\lambda = 9.1$, as shown in Table II. Assume that the cost of any surplus is $K = 500$, and the unit cost of shortage is $C_2 = 50$. From (4) we have as the condition for an optimum at I_o,

$$\frac{1 - F(I_o)}{p(I_o+1)} < 10 < \frac{1 - F(I_o-1)}{p(I_o)} .$$

TABLE II

A Poisson Distribution of Demand

$$P(D) = \frac{9.1^D e^{-9.1}}{D!} \qquad D=0,1,2,\ldots$$

D	P(D)	F(D)	D	P(D)	F(D)
0	.0001	.0001	12	.0752	.8683
1	.0010	.0011	13	.0526	.9209
2	.0046	.0057	14	.0342	.9551
3	.0140	.0197	15	.0208	.9759
4	.0319	.0516	16	.0118	.9877
5	.0581	.1097	17	.0063	.9940
6	.0881	.1978	18	.0032	.9972
7	.1145	.3123	19	.0015	.9987
8	.1302	.4425	20	.0007	.9994
9	.1317	.5742	21	.0003	.9997
10	.1198	.6940	22	.0001	.9998
11	.0991	.7931	23	.0000	.9998

We wish to find an optimal inventory level I_o which satisfies this condition. From Table II we find

$$\frac{1 - F(6)}{p(7)} = 7.0 ,$$

and

$$\frac{1 - F(5)}{p(6)} = 10.1 .$$

Therefore, we obtain

$$I_o = 6 .$$

B. MINIMIZING EXPECTED COST WHEN DEMAND IS CONTINUOUS

When the demand is continuous, the expected total cost function is shown by

$$E(I) = K \int_0^I p(D)dD + C_2 \int_I^\infty (D-I)p(D)dD$$

$$= K \int_0^I p(D)dD + C_2 \int_I^\infty Dp(D)dD - C_2 \int_I^\infty Ip(D)dD \; .$$

To find the optimal value I_o, we take the derivative with respect to I and set it equal to zero. Using Leibniz's rule, we obtain

$$\frac{dE(I)}{dI} = Kp(I) - C_2Ip(I) - C_2(1-F(I)) + C_2Ip(I)$$

$$= Kp(I) - C_2(1-F(I)) = 0 \; .$$

Then, we find that the optimal inventory level I_o is that which satisfies

$$F(I_o) = 1 - \frac{K}{C_2} p(I_o) \; , \qquad (5)$$

where $F(I) = \int_0^I p(D)dD$.

To illustrate this, suppose demand is normally distributed with $\mu = 10.0$ and $\sigma = 3.85$, and let $K = 500.0$ and $C_2 = 50.0$.

We seek a value for I_o which satisfies

$$F(I_o) = 1 - 10.0\, p(I_o) \quad ,$$

and using tables of the normal density and distribution functions, we find

$$I_o = 3.49 \quad .$$

C. ASPIRATION LEVEL SOLUTIONS

It is possibly true that some form of aspiration level principle is the most widely used of all principles of choice in management decision making [16]. An aspiration level is simply some level of cost which the decision maker desires not to exceed. For a decision under risk, an aspiration level policy might be expressed as follows. For a given aspiration level A, select the alternative which maximizes the probability that the cost will be equal to or less than A. Using this principle we shall find the optimal inventory level for our cost function in the following way.

We shall look first at the case where the aspiration level is greater than the surplus cost K. From Figure 1, we see that

$$\Pr(\text{cost} \leq A_1) = \Pr(0 \leq D \leq I + \frac{A_1}{C_2}) = F(I + \frac{A_1}{C_2})$$

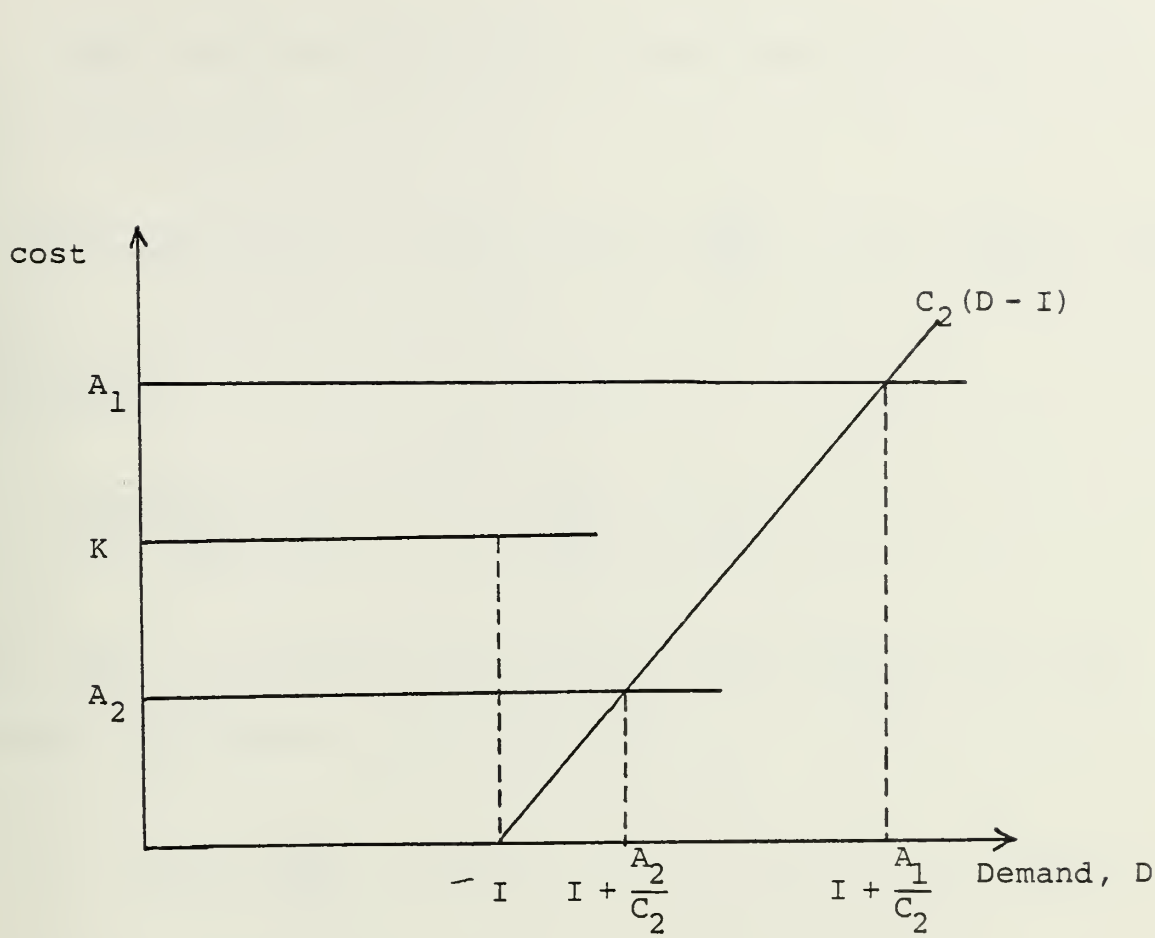

FIGURE 1. The Aspiration Level and Cost vs. Demand Relationship with Constant Costs for Surplus and Linear Costs for Shortage.

To find the optimal level I_o, we take the derivative with respect to I and set it equal to zero;

$$\frac{dPr(\text{cost} < A_1)}{dI} = f(I_o + \frac{A_1}{C_2}) = 0 \,.$$

If p(D) is unimodal, then I_o goes to infinity. This is clear from Figure 1 since the cost will be always $K < A_1$ as I goes to infinity. Thus we conclude that when the constant cost of surplus is less than our aspiration level, the optimal inventory policy is to stock as much as possible.

When the aspiration level is less than K, say $A_2 < K$, it follows that

$$\Pr(\text{cost} \le A_2) = \Pr(I \le D \le I + \frac{A_2}{C_2}) = F(I + \frac{A_2}{C_2}) - F(I) .$$

As before,

$$\frac{d\Pr(\text{cost} < A_2)}{dI} = f(I + \frac{A_2}{C_2}) - f(I) = 0$$

If $f(D)$ is unimodal and symmetric about $\bar{D}$ such as in the normal distribution, then the rule

$$f(I_o + \frac{A_2}{C_2}) = f(I_o) \qquad (6)$$

gives us optimal level I_o as

$$I_o = \bar{D} - \frac{1}{2}\frac{A_2}{C_2} .$$

This is shown in Figure 2.

D. SOLUTIONS UNDER UNCERTAINTY

We now consider the role of the density function of demand p(D) in this single-period inventory model. Clearly, in many situations it will be possible through the study of past demand, trends, and possible future contingencies, to form an estimate of the demand probability distribution.

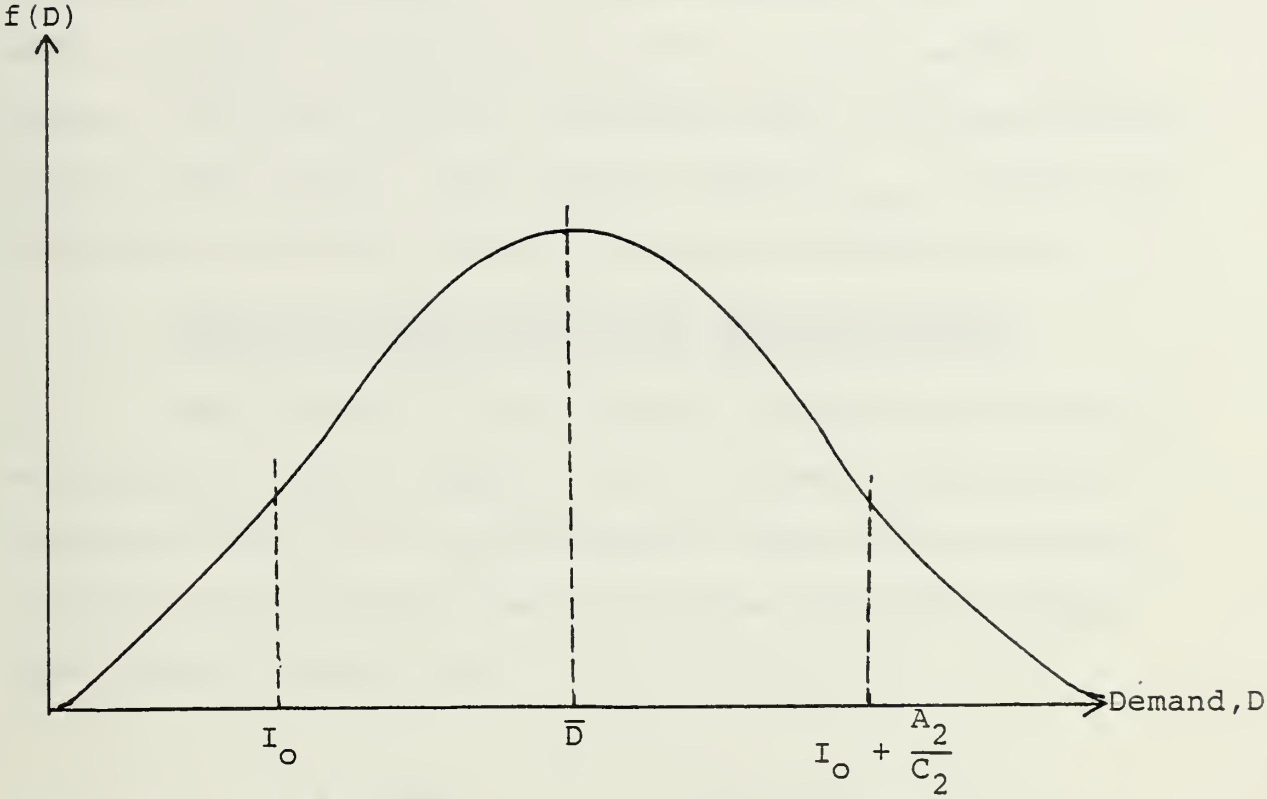

FIGURE 2. Application of the Aspiration Level Decision Rule to a Unimodal, Symmetric Probability Distribution for Demand.

However, in some situations this may not be possible. For example, when a buyer is going to buy new merchandise, he must estimate the future demand for the merchandise. Since the merchandise is new, he will probably not know this beforehand. Demand is influenced by many uncertain factors including the appeal of the particular products, the trends in style, general economic conditions and other factors, and thus no reasonable basis for prediction exists. The

buying decision is thus a decision under uncertainty. [14] Actually, the decision maker may be only "slightly ignorant" of the demand. It will be assumed that he is able to predict a finite minimum and a finite maximum for demand, and nothing more. We assume that the minimum demand is zero and we shall call maximum demand D_{max}. We show two alternative solutions under conditions of uncertainty.

1. Solutions Under Uncertainty when the Demand Probability Distribution is Assumed Uniform

When we fail to get a demand distribution function, one approach simply suggests that the density function of demand be taken as a uniform density function, and we then minimize total expected costs. [14] We set up the demand distribution function as

$$p(D) = \begin{cases} \dfrac{1}{D_{max}} & 0 \leq D \leq D_{max} , \\ 0 & \text{otherwise} . \end{cases}$$

Hence $$F(I_o) = \int_0^{I_o} p(D)dD = \int_0^{I_o} \frac{dD}{D_{max}} = \frac{I_o}{D_{max}} .$$

Substituting this value in Equation (5), we obtain the optimal inventory level I_o as

$$I_o = D_{max} - \frac{K}{C_2} \quad \text{when} \quad C_2 D_{max} > K ,$$

and see that I_o is obtained by subtracting the ratio of the constant surplus cost K and the unit shortage cost C_2 from estimated maximum demand.

2. <u>Minimax Solution</u>

We shall now look for a minimax cost solution. In the cost expression

$$\text{Cost} = \begin{cases} K & D < I \; , \\ C_2(D - I) & I \leq D \; , \end{cases}$$

the maximum cost is a constant when demand is less than inventory. When demand is greater than inventory, $C_2(D - I)$ takes its maximum where D is D_{max} and I is zero, thus

$$C_2(D - I) \quad = \quad C_2 D_{max} \qquad \text{where } I = 0 \qquad .$$

We must consider whether C_2D_{max} is greater than K or less than K.

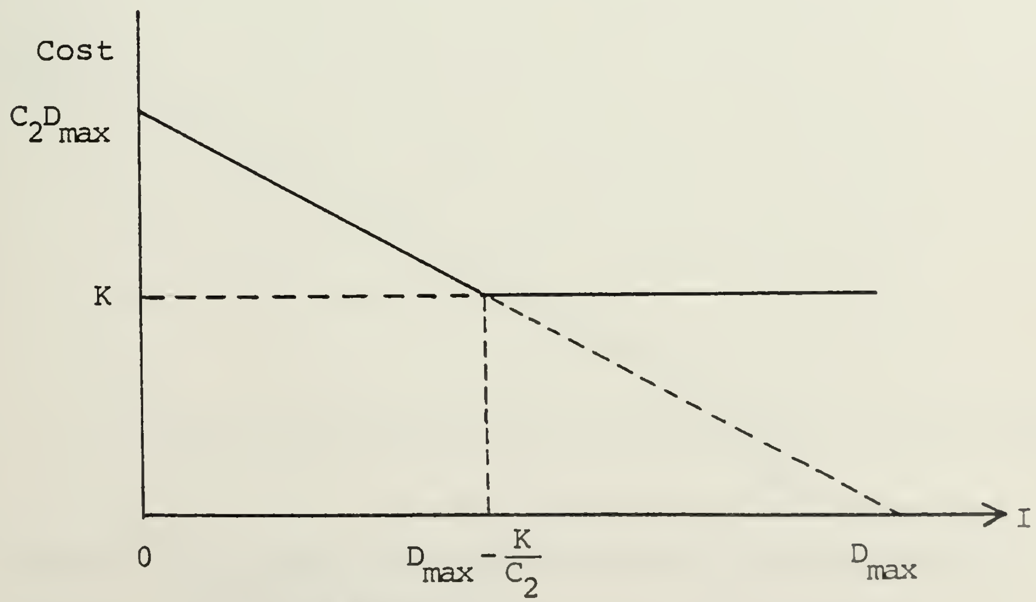

FIGURE 3. Maximum Cost Line Against Inventory Level When C_2D_{max} > K.

We can see from Figure 3 that the minimum cost occurs at the intersection of the two functions, where

$$K = C_2(D_{max} - I) \quad .$$

Thus we have

$$I = D_{max} - \frac{K}{C_2} \quad .$$

It is clear that the inventory level cannot exceed D_{max}. Thus, the inventory level is

$$D_{max} > I_o > D_{max} - \frac{K}{C_2} \quad \text{when} \quad C_2D_{max} > K \; .$$

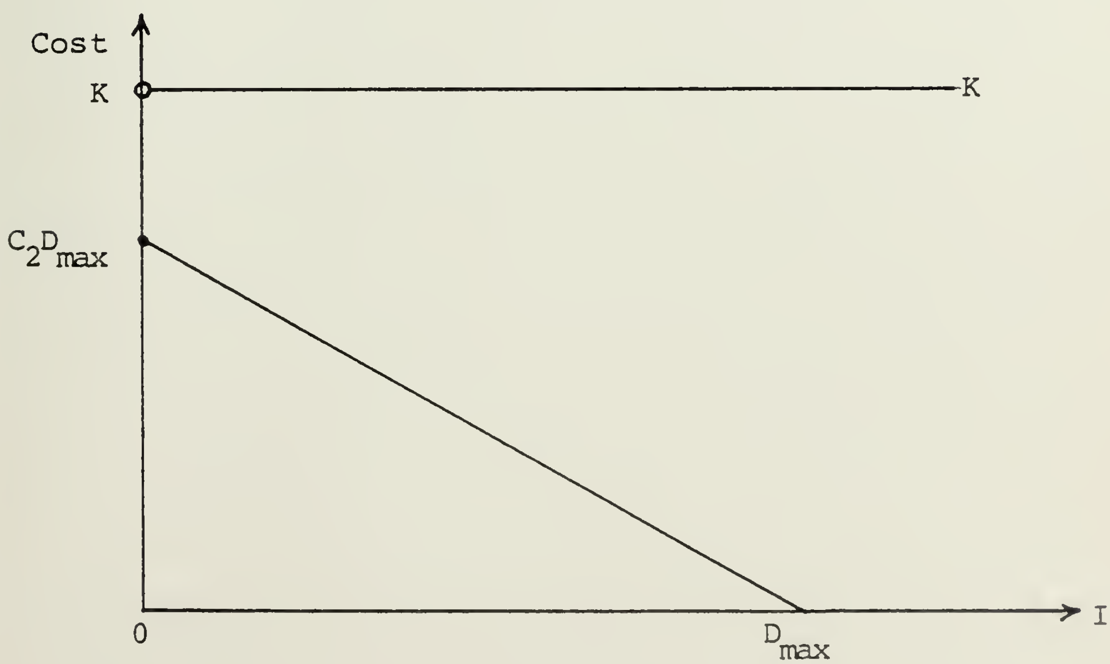

FIGURE 4. Maximum Cost Line Against Inventory Level When $C_2D_{max} < K$

We can see from Figure 4 that the minimum occurs at $I = 0$. Thus the optimal inventory level for this case is

$$I_o = 0 \qquad \text{when} \quad C_2 D_{max} < K \quad .$$

In the next chapter, we shall look at real world applications for the classic newspaper boy problem and its extensions.

V. APPLICATIONS IN THE REAL WORLD OF THE SINGLE-PERIOD INVENTORY MODEL

In earlier chapters, we have discussed the historical development of the single-period inventory problem and shown extensions of the analytic solution for a constant surplus cost. We will also show analytic solutions for various other kinds of cost functions in Appendix A and Appendix B. We will now look at how these results have been applied in the real world, as well as at interesting suggested applications.

From the literature, there is evidence of widespread applications in business and industry. [6],[17] Applications to military inventory problems show success as well. Here, the elementary single-period models with which we are concerned also have helped to clarify a portion of the inventory problem. From a practical viewpoint, the need for specific data is vital for the successful application of these inventory models. Because of this, the major portion of most studies is devoted to determination of set-up cost, inventory carrying cost, shortage cost and demand patterns.

First of all, we will discuss the demand function. Then, we will introduce various kinds of examples from the literature representing real or anticipated applications for the newspaper boy problem. Many of these especially

concern shortage cost and overage cost. We will first look at military applications, and then proceed with applications for perishable items, seasonal items and others.

A. SOME CONSIDERATIONS FOR THE DEMAND FUNCTION

The problem of forecasting demand is very important in order to obtain the optimal inventory level in the newspaper boy problem. If we wish a functional form for the demand distribution, then we will seek the probability function that best fits demand for a given product. Generally, we collect past demand data such as number of customers per day, number of units demanded per customer, or number of units demanded per month. The data for these random variables may then be checked against known analytic probability functions by means of a chi-square goodness-of-fit test or other statistical technique.

As we discussed before, we may have situations where there is insufficient data to fit possible future demand probability functions. For example, it is very hard to know the demand for new merchandise, and for this case, we have already introduced two methods: setting up a uniform density function as the demand function, or using the minimax cost solution.

There has been some effort to define rules for estimating probability functions for demand. D.A. Shrady mentions guidelines for selecting demand distributions for use in military supply systems [18]:

The distribution and moments of leadtime demand are the central parameters of any inventory policy. In UICP (Uniform Inventory Control Procedures) the distribution of lead time demand is assumed to be Poisson if the mean annual demand is less than or equal to 1. If the mean annual demand is 10 or greater, a normal distribution is used. If the mean annual demand is between 1 and 10, the negative binomial distribuiton is used.

Another approach for single-period distributions is introduced by Giffin. [19]:

A passenger ship about to sail from San Francisco to Honolulu can be expected to require a fixed number of days to make the trip. A single-period inventory model might be used to determine the stores of perishable fruit to be served on board. From past experience it may be possible to determine that the daily consumption by passengers is distributed by f(x). Furthermore it is known that there are n passengers on board and that the trip will require d days.

Total demand per day is the sum of n independent random variables x_i, each distributed by $f(x_i)$

$$\text{Daily demand} = x_1 + x_2 + \ldots + x_n$$

The probability distribution g(y) for daily demand y is the n-fold convolution of f(x). Since the convolution operation reduces to a simple multiplication of transforms, we have the transform of the daily demand distribution given by

$$T[g(y)] = [T(f(x))]^n$$

Total demand for the trip is the sum of d independent random variables y_i each distributed by $g(y_i)$. Here y_i represents total consumption for day i.

$$\text{Trip demand} = t = y_1 + y_2 + \ldots + y_d$$

The transform of the probability distribution h(t) representing total trip demand is then

$$T[h(t)] = [T(g(y))]^d = [T(f(x))]^{nd}$$

The distribution can be obtained by inverting $T[h(t)]$. If inversion is impossible it can be approximated by determining an appropriate number of moments which can be used to establish the parameters of some general class of distribution such as gamma.

Yet another approach would employ the distribution estimation procedures from PERT. [20] Here the demand distribution is assumed to be Beta, and we estimate it as follows.

D_{min} = smallest demand value — the unlikely but possible minimum demand.

D_{max} = largest demand value — the unlikely but possible maximum demand.

m = most likely demand — the value which is likely to occur more often than any other value.

Note that m will be an estimate of the mode of distribution of the task time, rather than the mean.

Demands are assumed to follow the Beta Distribution with $D_{min} \leq D \leq D_{max}$ and mode m. Approximations to the mean and variance of demand are taken as:

$$E(D) = \frac{D_{min} + 4m + D_{max}}{6} , \tag{7}$$

$$Var(D) = \left(\frac{D_{max} - D_{min}}{6}\right)^2 . \tag{8}$$

Using the estimates D_{min}, D_{max}, and m, estimates of the mean and variance may be computed by the approximations (7) and (8).

B. MILITARY APPLICATIONS

When a shortage occurs for a military item, the problem is somewhat different from that of the private entrepreneur. For example, if a resident of Monterey goes shopping in a department store and finds the store is out of the article he desires, his demand probably can be satisfied in some other store. Such a case involves very little harm. The first store has probably lost some profits and some good will and nothing else. Although the military situation may seem conceptually similar to that of the private entrepreneur, the difference in costs may be tremendous, since if important items of equipment are not available when needed as in war-time the fate of the nation may be in danger. Even if the equipment is available, it may be at some other base and is useless because of the lack of immediate accessibility. [7]

The following three examples are related to military applications for spare parts. A piece of equipment produced for the military has the characteristic that the spare parts which may be needed during its lifetime are produced at the same time as the equipment itself. If not enough spares are produced, it may be extremely costly or impossible to obtain them at a later date. There then arises the question of how many spares of each type of parts should be produced.
The number of spares to be demanded over the lifetime of a piece of equipment cannot be predicted with certainty, but must instead be treated as a random variable. Consider then a typical example of such a situation.

> 1. A new ship is being constructed for the Navy which will have a lifetime of about 20 years. The Navy wishes to specify how many spares of a particular very expensive part should be produced when the ship is being constructed. The part might be the shaft and fittings for the screw, for example, or perhaps the rudder mechanism [21].
>
> 2. A certain military office is buying an experimental weapon system. Spare parts are most easily manufactured at the time of manufacture of the prototype. They are expensive to obtain later. Any unused units left at the end of the test period can be sold as surplus at a lower price. The number of treads worn out is related to the length of test period and is given by past experience. Therefore the problem is how many spare parts should be procured? [19]
>
> 3. Let us suppose that each part will cost $50,000 when produced simultaneously with the ship, but $150,000 if it must be produced after the ship is launched. After a study of similar ships, the Navy decides that no more than six of these spares should ever be needed, and it estimates the probabilities of various numbers of parts being required. Any parts left over when the ship is scrapped will have a salvage value of $1000 each. From this information, we would like to determine how many spares should be ordered. [21]

There are similar applications with other systems such as aircraft, tanks, missiles, radar and so on. [5]

Swasdikiat introduced a good military application using a linear function for surplus cost and a constant for shortage cost. [15]

> This kind of cost function might arise in such problems as determining the number of bombs for bombers to carry in one flying mission. An excess number of bombs will cost the Air Force C_1 dollars each. A shortage of bombs, any number at all, will cause the mission to be incomplete, this will cost the Air Force K dollars. A similar example is the number of shells for an artillery battery to take with them from the ammunition dump. An excess number will cost the

battery C_1 dollars each. The shortage of shells, any number at all, will cost the battery commander to send a truck back to the ammunition dump to get them, which will cost the battery K dollars.

His analytical solution for this problem is shown in Appendix A.

C. APPLICATIONS FOR PERISHABLE ITEMS

Problems associated with management of perishable inventories arise in many areas, and they have the following characteristics. Buying perishable items for the day, the manager will see them spoil if he buys too many or will turn customers away unsatisfied if he buys too few, thus failing to earn potential profits and perhaps permanently losing some customers. The supermarket and food industry are characterized by many fixed-life commodities.

Another important example relates to health care delivery in the form of blood bank management since in most areas, the shelf life of whole blood is limited to 21 days. Similarly, many drugs must be discarded after a fixed period.

A discussion of general perishable applications is given in ORSA, Management Science, and Naval Research Logistic Quarterly articles written by Nahmias. [22],[23],[24] Specific applications and suggested applications in the literature include the following.

1. A large supermarket must decide how much bread or cake (or other perishable item) to purchase each day considering past demand history for the particular day of week. [5]

2. The concessionare at the local football stadium is faced with the decision of how many cups of hot chocolate to prepare before a football game. All leftover chocolate is disposed of at a complete loss. [19]

Similar problems concerning perishable items (such as meat, vegetables, eggs, and so on) and the newspaper boy problem appear in inventory books.

D. APPLICATIONS FOR SEASONAL ITEMS

Seasonal variations are regular rhythmic movements within a period of one year resulting from the weather and from man-made conventions such as holidays or the fiscal year. Regular rhythms in demand also occur within a quarterly, monthly, weekly, or daily period. Department store sales, for example, expand before Easter and Christmas (a circumstance related to man-made festivals) and there may be losses at the end of season for left over goods. The Christmas tree problem, of course, is one of the best-known newspaper boy applications for seasonal items. Let us look at some other examples:

1. A dressmaker is interested in buying a special type of imported fabric for manufacture for the coming one week fiesta season. The fabric must be ordered well in advance of the fiesta. Any fabric remaining in stock at the end of the season can be returned at a lower price than the purchase price, while finished dresses can be sold after the season at sales. The problem, of course, is to determine the optimal number of (dress) units of fabric to purchase. [5]

2. Consider a department store which must stock bathing suits for the summer season. A particular model suit costs the company $5 and normally sells for $9. Any suits left at the end of the season are discounted to $3 each to clear the shelves. Management estimates that a goodwill cost of $2 is incurred when they are unable to supply the suit a customer desires. Historical data from past years indicate that sales for the season are approximately normally distributed with mean 400 and standard deviation 100. [19]

3. In the case of a fashionable candy store, the store must decide how many large chocolate rabbits to order for the Easter season two months in advance. There is no possibility of placing a reorder. Any rabbits not sold at the end of the season are a total loss. [5]

4. The buyer of a large West Coast department store must decide what quantity of a high-priced, woman's leather handbag to procure in Italy for the coming Christmas season. [5]

5. A bookstore is about to stock a sensational new book written by a controversial person who is currently in the news. After a short selling season, the book will be stale, and it will have no salvage value. The problem is to determine how many should be stocked under the certain anticipated distribution. [25]

6. The Tofer Swimming Pool Company sells a particular model for $500. It costs the company $300, and its salvage price is $200 (all residual sale stock is sold in the August Sale at $200 per unit). The manager must decide how many pools to stock for the coming season, knowing the probability distribution of demand. [25]

E. OTHER APPLICATIONS

The application of inventory control methods to style goods is also of great importance. For example:

1. In the mail order business the control buyers make important decisions concerning what quantities of the various items should be bought. These decisions to a large extent determine whether the supply of each item is less than, equal to, or greater than the demand. When supply is less than demand, a customer's orders cannot be filled, making letters of apology and refunds necessary. When supply is greater than demand, the excess usually must be liquidated at a loss. When supply is less than demand, an "omission" is said to occur; when supply is greater than demand, a "surplus". [1]

Another interesting application concerns a school run for airline hostesses. [26] This is not exactly a single-period problem since we consider the effects of an appreciable lead time between the successive decisions to produce for inventory. Nevertheless, this is a sort of newspaper boy problem and really happened in the real world. The problem is:

2. An airline runs a school for air hostesses each month; it takes to months to assemble a group of girls and to train them. Past records of turnover in hostesses show that the probability of requiring x new trained hostesses in any one month is $g(x)$ $[x = 0,1,2,\ldots]$, and the probability of requiring y new hostesses in any two-month period is $h(y)$. In the event that a trained hostess is not required for flying duties, the airline still has to pay her salary at the rate of C_1 per month. If insufficient hostesses are available, there is a cost of C_2 per girl short per month. We seek decision rules for the size of classes.

The next application relates to a Navy Exchange Uniform shop.

3. The Naval Postgraduate School has an annual Military Ball in October. The exchange must order uniforms for rent, which are shipped from the East Coast. Customers make reservations in advance, and the exchange may order additional

uniforms for people who want to rent but did not make reservations. This could be solved as a newspaper boy problem to decide how many uniforms should be ordered of a certain type.

Similarly, yearly events such as the next annual banquet generate problems which are also appropriate for solution as newspaper boy problems.

4. In planning a certain annual alumni banquet, the banquet manager informs us that we must specify within the next few days the number we expect to attend. He gives us a price of $6 per plate for the exact number specified. Additional dinners may be obtained on the day of the banquet at a price of $8 each. If fewer dinners are needed than ordered, a partial refund of $2 will be made for each dinner not needed: the fee that we will charge the alumni has been set at $10 each for those attending. Because of the short time available it is not possible to use a mail reservation system. [27]

The next application concerns Saki which a blackmarket dealer sells to soldiers during the war.

5. Tri Tu Chetime is a blackmarket dealer in a small hamlet in South We-et-napalm. He has heard rumors that a large U.S. offensive is scheduled to end in his village. He is interested in selling Rot Saki to the G.I.s at $5 per bottle. Each bottle costs him $1, and he knows that any bottles left after the soldiers depart may be sold to the resident VC at $0.5 per bottle. Tri fears that if he runs out of Saki, the soldiers will be unhappy and will trample his rice paddies. He estimates this cost at $10 per bottle. His local advisor for Advanced Inventory Divisions (AID) has estimated that G.I. demand will be normally distributed with a mean of 900 bottles and variance of 9×10^4. How many bottles should Tri purchase? [19]

The following example is for an electric power company, and is similar to the military spares example given earlier.

6. An electric power company is about to order a new generator for its plant. One of the essential parts of the generator is very complicated and expensive and would be impractical to order except with the order of generators. Each of these spare parts is uniquely built for a particular generator and may not be used on any other. The company wants to know how many spare parts should be incorporated in the order for each generator. [28]

Other examples are

7. The Zip Car Rental Company rents cars at a rate of $10 per day. Cars are rented for one day only. Zip Company does not own its own cars but leases them on a daily basis from a large leasing firm. The larger firm pays the maintenance cost for the cars. Zip must specify the number of cars it intends to lease on a given day at least one week in advance. The daily lease fee paid to the leasing firm by Zip Company is $7 per day. Zip is faced with the decision of how many cars to lease for a given day one week hence. The demand for rental cars varies from day to day. If Zip Company leases more cars than are requested as rentals on a particular day, Zip Company will lose the lease fee of $7 for each car unrented. If demand for cars is greater than the number available, a profit of $3 per car is foregone. [27]

8. The Fox photo company is a mail order firm specializing in 24-hour service on developing negatives and making prints. The general policy is that orders arriving in the morning mail must be finished and in the outgoing mail before the midnight mail pickup. This has usually involved little difficulty. Six full-time technical people work an eight-hour day from 8 A.M. to 5 P.M. and are paid at a rate of $4 per hour (including fringe benefits). These technicians can process an average of 5 orders an hour. When, on occasion, more than about 240 orders arrive in a given day, one or more of the men works overtime at a rate of $6 per hour. Fox Photo has recently bought out a competitor in the same community and plans to consolidate operations. Mr. Fox is undecided, however, on how many technicians to add to the six he now employs. By adding together the past order data of his competitor to his own, Mr. Fox has the past frequency data, and must determine how many additional technicians to employ. [28]

Finally, a most interesting account of a study of the parking space requirements for automobiles at airports was described in 1955 by Hurst of the Port of New York Authority, which is responsible for the operation of the three major airports in the New York area. The treatment is based on a inventory type model which involves shortage and overage costs. [29] We shall describe this application in some detail. The situation is as follows:

> The Port Authority manages the three major airports in the New York-New Jersey Metropolitan Region. Inherent in this function is the responsibility for anticipating as early as possible when existing facilities will become obsolete. The sum total of a number of these evaluations yields a development program. As a consequence of such programs, an 8-million dollar terminal building was opened at New York Airport in August 1953, and plans for 60-million dollars' worth of new terminal and hangar facilities at New York International Airport are now being implemented. The problem is how many parking spaces are required.
>
> These estimates of parking-lot requirements are, of course, strongly related to the evaluations made or being made for other facets of the airport economy. There are public parking facilities provided at all three airports. These facilities are provided and administered by the Port Authority. The actual operation, however, is in the hands of concessionnaires. Daily reports of parking-lot operations are made to the Port Authority by the concessionnaires. These reports give detailed financial and operational characteristics, such as the number of cars paying a fee and amount of the fee. The characteristics of employee parking are entirely separable from those of the public parking.

Summarization and some considerations for this problem are as follows:

> In evaluating the adequacy of airport lots for a given future year, it is necessary to forecast the demand for parking spaces for that year. As in all inventory problems, the forecasting error is critical. With accurate knowledge of demand, the optimal inventory level (i.e., the size of the

parking lot) would be trivially known: it would be equal to the known demand. With probabilistic demand, however, the optimal size of the lot can only be defined as that size which minimizes total expected cost of overage and shortage spaces for the year under consideration. If this optimal size falls below the actual size, the lot is still adequate.

The cost of shortage in this application was taken to consist of lost parking-lot revenues, airport revenues, and administrative cost of responding to bad will created among displaced parking-lot patrons. The cost of overage is the cost of building and operating an excess of spaces. These costs could be determined. The forecast of demand for parking spaces was based on forecasts of air traffic and related traffic magnitudes and patterns.

In the next chapter, we shall describe suggested further work and conclusions.

VI. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

We began this thesis by tracing the original person who derived the powerful analytic solution for the newspaper boy problem, finding that Whitin and Youngs derived the solution although Morse and Kimball gave the idea for the newspaper boy problem originally. We reviewed the standard single-period inventory problem in Chapter III, showing the aspiration level and minimax solutions. We developed procedures for finding the optimal inventory level when there is a constant as surplus cost and linear function as shortage cost. In Chapter V, we stated some considerations for the demand function and reviewed various kinds of applications for military items, perishable items, seasonal items, and others.

The author realized that finding the demand function is one of the important key points to obtain the optimal inventory level in the real world. Therefore, study of demand functions and ways of estimating them in more detail is recommended as further work.

For some cost functions, it is sometimes difficult to express the optimal inventory level I_o in terms of tractable expressions, and the general optimal solution is left in integral form as in Appendices A and B. Finding approximations for the general solutions or exact solutions for classes of demand functions is also suggested as further work.

It is hoped that the work contained in this thesis will assist and be of use to readers who are interested in this kind of inventory system.

APPENDIX A

SUMMARY OF THREE VARIATIONS ON THE SINGLE-PERIOD MODEL

In this appendix, we simply state results for three variations on the single-period model's cost function.

1. The cost equation with quadratic cost function is [15]

$$\text{cost} = \begin{cases} C_1(I - D)^2 \quad , \quad D = 0,1,2,\ldots,I \\ C_2(D - I)^2 \quad , \quad D = I+1,\ldots,D_{max} \; . \end{cases}$$

(a) Discrete Case Solution

$$-(C_1 - C_2)F(I_o) + C_2 < 2 \sum_{D=0}^{I_o} C_1(I_o - D)p(D) - \sum_{D=I_o+1}^{\infty} C_2(D - I_o)P(D))$$

$$< (C_1 - C_2)F(I_o - 1) + C_2 \; .$$

(b) Continuous Case Solution

$$F(I_o) = \frac{C_1 + C_2}{I_o(C_1 - C_2)} \int_{D=0}^{I_o} Dp(D)dD - \frac{C_2}{I_o(C_1 - C_2)} (I_o - E(D)) \; .$$

2. The cost equation with a fixed shortage cost is [15]

$$\text{cost} = \begin{cases} C_1(I - D) & , \quad D = 0,1,2,\ldots,I \\ K & , \quad D = I+1,\ldots,D_{max} . \end{cases}$$

(a) <u>Discrete Case Solution</u>

$$\frac{F(I_o - 1)}{p(I_o)} < \frac{K}{C_1} < \frac{F(I_o)}{p(I_o + 1)} .$$

(b) <u>Continuous Case Solution</u>

$$F(I_o) = \frac{K}{C_1} p(I_o) .$$

3. The cost equation with a quadratic shortage cost is [16]

$$\text{cost} = \begin{cases} C_1(I - D) & , \quad D = 0,1,2,\ldots,I \\ C_2(D - I)^2 & , \quad D = I+1,I+2,\ldots,D_{max} . \end{cases}$$

(a) <u>Discrete Case Solution</u>

$$L(I_o) \leq C_1/2C_2 \leq U(I_o) ,$$

where

$$L(I_o) = (\overline{D}^c(I_o) + F^c(I_o)(I_o - \tfrac{1}{2}))/F(I_o) ,$$

$$U(I_o) = (\overline{D}^c(I_o) + F^c(I_o)(I_o + \tfrac{1}{2}))/F(I_o) ,$$

$$\overline{D}^c(I_o) = \sum_{D=I_o+1}^{\infty} Dp(D) ,$$

and

$$F^c(I_o) = \sum_{D=I_o+1}^{\infty} p(D) .$$

(b) <u>Continuous Case Solution</u>

$$\frac{C_1}{2C_2} = (\overline{D}^c(I_o) - I_o F^c(I_o))/F(I_o) ,$$

where

$$\overline{D}^c(I_o) = \int_{I_o}^{\infty} Dp(D)dD$$

and

$$F^c(I_o) = \int_{I_o}^{\infty} p(D)dD .$$

APPENDIX B

THE SINGLE-PERIOD MODEL WITH EXPONENTIAL COST FUNCTIONS

In this appendix, we shall show the derivation of the expected value solution when the cost function is exponential and demand is a continuous random variable. The cost equation for the exponential case is

$$\text{cost} = \begin{cases} e^{-C_1(I - D)} & I \geq D \; , \\ e^{-C_2(D - I)} & D > I \; . \end{cases}$$

The total expected cost equation is

$$E(I) = \int_0^I e^{-C_1(I-D)} p(D) + \int_I^\infty e^{-C_2(D-I)} p(D)\, dD \; .$$

Taking the derivative and setting it equal to zero, we have

$$\frac{dE(I)}{dI} = -C_1 e^{-C_1 I} \int_0^I e^{C_1 D} p(D)dD + e^{-C_1 I} e^{C_1 I} p(I)$$

$$+ C_2 e^{-C_2 I} \int_I^\infty e^{-C_2 D} p(D)dD - e^{C_2 I} e^{-C_2 I} p(I) = 0 \; .$$

Therefore the optimal inventory level I_o must satisfy the following equation:

$$C_1 e^{-C_1 I_o} \int_0^{I_o} e^{C_1 D} p(D)\,dD = C_2 e^{C_2 I_o} \int_{I_o}^{\infty} e^{-C_2 D} p(D)\,dD \;. \qquad (9)$$

Suppose the demand function is the exponential density function with parameter λ . We substitute $e^{-\lambda D}$ into (9):

$$C_1 e^{-C_1 I_o} \int_0^{I_o} e^{D(C_1-\lambda)}\,dD = C_2 e^{C_2 I_o} \int_{I_o}^{\infty} e^{-D(C_2+\lambda)} \;,$$

and obtain

$$I_o = \frac{1}{\lambda + C_2} \ln \left\{ 1 - \frac{C_1 (C_2 + \lambda)}{C_2 (C_1 - \lambda)} \right\} .$$

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